Momentum Integral Theorems for the Boundary layer:

Solution of the boundary layer Equations

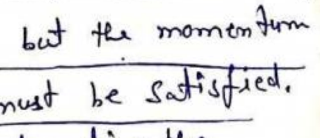
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{—— (1) (i)}$$

$$\frac{\partial p}{\partial y} = 0 \quad \text{———— 1 (ii)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{———— 1 (iii)}$$

is in general very difficult. The Sol^{n}s obtained previously is ~~are~~ very special case. For Engineering problems, it is often acceptable if an approximate Solution can be obtained. one of the most useful methods is the Von Karman - Pohlhousen method based on the Integral theorem. The basic Concept of this method is that the sol^n's satisfy the diff. eq^{n}s only on the average. In other words, it is not anticipated that the Sol^n satisfies the boundary layer differtial Eq^{n}s at every point (x,y) but the momentum integral Eq^n and the boundary Conditions must be satisfied. The momentum Integral Eq^n is obtained by integrating the boundary layer Equations w.r.t y.

The Von Karman - Pohlhausen / Integral Relation:



Integrating Eq^n 1(i) w.r.t y from y=0 to y=δ(x) we have

$$\int_0^{\delta}\left(\frac{\partial u}{\partial t}\right)dy + \int_0^{\delta} u\frac{\partial u}{\partial x}\,dy + \int_0^{\delta} v\,\frac{\partial u}{\partial y}\,dy = -\frac{1}{\rho}\int_0^{\delta}\frac{dP}{dx}\,dy$$

$$+\frac{\mu}{\rho}\int_0^{\delta}\frac{\partial^2 u}{\partial y^2}\,dy$$

$$\int_0^\delta \left(\frac{\partial u}{\partial t}\right) dy + \int_0^\delta \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) dy = -\frac{1}{\rho} \int_0^\delta \frac{dp}{dx}\, dy + \frac{\mu}{\rho}\int_0^\delta \frac{\partial^2 u}{\partial y^2}\, dy \quad \text{—(2)}$$

now,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u\frac{\partial u}{\partial x} + \frac{\partial(uv)}{\partial y} - u\frac{\partial v}{\partial y}$$

$$= u\frac{\partial u}{\partial x} + \frac{\partial(uv)}{\partial y} + u\frac{\partial u}{\partial x} \qquad \text{(using Eq}^n \text{ 1(iii))}$$

$$= 2u\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}(uv)$$

$$= \frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y}$$

using this relation Eq^n (2) ~~reduces to~~ gives us

$$\int_0^\delta \left(\frac{\partial u}{\partial t}\right) dy + \int_0^\delta \left\{\frac{\partial(u^2)}{\partial x} + \frac{\partial(uv)}{\partial y}\right\} dy = -\frac{1}{\rho}\frac{dp}{dx}\cdot\delta + \frac{\mu}{\rho}\left\{\left(\frac{\partial u}{\partial y}\right)_\delta - \left(\frac{\partial u}{\partial y}\right)_0\right\}$$

$$\int_0^\delta \left(\frac{\partial u}{\partial t}\right) dy + \int_0^\delta \frac{\partial}{\partial x}(u^2)\, dy + [uv]_0^\delta = -\frac{1}{\rho}\frac{\partial p}{\partial x}\delta + \frac{\mu}{\rho}\left\{\left(\frac{\partial u}{\partial y}\right)_\delta - \left(\frac{\partial u}{\partial y}\right)_0\right\} \quad \text{—(3)}$$

But if $u = U$ when $y = \delta$

$$[uv]_0^\delta = U.v_\delta = U\int_0^\delta \left(\frac{\partial v}{\partial y}\right) dy = U\int_0^\delta \left(-\frac{\partial u}{\partial x}\right) dy$$

also if $\left(\frac{\partial u}{\partial y}\right) = 0$ when $y = \delta$, So Eq^n (3) reduce to

$$\int_0^\delta \left(\frac{\partial u}{\partial t}\right) dy + \int_0^\delta \frac{\partial}{\partial x}(u^2)\, dy - U\int_0^\delta \left(\frac{\partial u}{\partial x}\right) dy = -\frac{\delta}{\rho}\frac{\partial p}{\partial x} - \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)_0 \quad \text{—(4)}$$

now using Leibnitz rule of integration we get

$$\frac{d}{dx}\int_0^\delta u^2\, dy = \int_0^\delta \frac{\partial u^2}{\partial x}\, dy + U^2\frac{d\delta}{dx}.$$

$$\Rightarrow \int_0^\delta \frac{\partial u^2}{\partial x}\, dy = \frac{d}{dx}\int_0^\delta u^2\, dy - U^2\frac{d\delta}{dx}$$

and $$\frac{d}{dx}\int_0^\delta u\,dy = \int_0^\delta \frac{\partial u}{\partial x}\,dy + U\frac{d\delta}{dx}$$

multiply by U both side

$$U\cdot\frac{d}{dx}\int_0^\delta u\,dy = U\int_0^\delta\left(\frac{\partial u}{\partial x}\right)dy + U^2\frac{d\delta}{dx}$$

$$U\int_0^\delta\left(\frac{\partial u}{\partial x}\right)dy = U\frac{d}{dx}\int_0^\delta u\,dy - U^2\frac{d\delta}{dx}$$

and $$\frac{\partial}{\partial t}\int_0^\delta u\,dy = \int_0^\delta \frac{\partial u}{\partial t}\,dy + \cancel{\frac{\partial\delta}{\partial t}}^{\,0}\cdot U = \int_0^\delta \frac{\partial u}{\partial t}\,dy$$

$(\because \delta(x))$

Therefore Eqn (4) reduces to

$$\frac{\partial}{\partial t}\int_0^\delta\left(\frac{\partial u}{\partial t}\right)dy + \frac{d}{dx}\int_0^\delta u^2\,dy - \cancel{U^2\frac{d\delta}{dx}} - U\frac{d}{dx}\int_0^\delta u\,dy + \cancel{U^2\frac{d\delta}{dx}}$$
$$= -\frac{\delta}{\rho}\frac{dp}{dx} - \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$\boxed{\frac{\partial}{\partial t}\int_0^\delta u\,dy + \frac{d}{dx}\int_0^\delta u^2\,dy - U\frac{d}{dx}\int_0^\delta u\,dy = -\frac{\delta}{\rho}\frac{dp}{dx} - \frac{1}{\rho}\tau_0}$$

where τ_0 is shearing stress at the surface wall.

This is one form of the Von Karman integral relation, it is also known as momentum integral Equation of the boundary layer. ~~for steady flow first term at both side of the Eqn becomes zero.~~

Other Forms of the Von Karman Integral Relation :

It is often convenient to have the Integral relation in a form in terms of the displacement and momentum thicknesses for steady flow

we know by Euler Eqn of motion outside the boundary layer

$$U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{dx}$$

So, Von Karman Integral Eqn becomes

$$\frac{d}{dx}\int_0^\delta u^2 dy - U\frac{d}{dx}\int_0^\delta u\,dy - \frac{dU}{dx}\cdot U\delta = -\frac{\tau_0}{\rho}$$

or, $$\frac{d}{dx}\int_0^\delta u^2 dy - \frac{d}{dx}\left(\int_0^\delta Uu\,dy\right) + \left(\frac{dU}{dx}\right)\int_0^\delta u\,dy - \frac{dU}{dx}\int_0^\delta U dy = \frac{-\tau_0}{\rho}$$

or, $$\frac{d}{dx}\int_0^\delta u(U-u)\,dy + \frac{dU}{dx}\int_0^\delta (U-u)\,dy = \frac{\tau_0}{\rho}$$

taking limit $\delta \to \infty$ we get

$$\frac{d}{dx}\int_0^\infty u(U-u)dy + \frac{dU}{dx}\int_0^\infty (U-u)\,dy = \frac{\tau_0}{\rho}$$

$$\boxed{\frac{d}{dx}(U^2\delta_2) + \frac{dU}{dx}\cdot U\delta_1 = \frac{\tau_0}{\rho}} \Rightarrow \boxed{\frac{d\delta_2}{dx} + (2\delta_2+\delta_1)\frac{1}{U}\frac{dU}{dx} = \frac{\tau_0}{\rho U^2}}$$

where δ_1 & δ_2 are displacement and momentum thickness. given by $\delta_1 = \int_0^\infty \left(1-\frac{u}{U}\right)dy$

& $\delta_2 = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy$

Example:

For flow over a flat plate, the velocity profile within the boundary layer is assumed as $u = U \tanh\left(\frac{y}{\delta}\right)$. Show that the skin friction coefficient on the flat plate is given by:

0.3125

$$C_f = \frac{\tau_0}{\rho U^2} = \left(\cancel{0.392}\right)\left(\frac{Ux}{\nu}\right)^{-\frac{1}{2}}$$

where symbols have their usual meanings.

Sol^n: The von Karman integral relation is

$$\frac{d}{dx}\int_0^\delta u(U-u)\,dy + \frac{dU}{dx}\cdot\int_0^\delta (U-u)\,dy = \frac{\tau_0}{\rho}$$

here $U =$ const. So above eq^n reduces to

$$\frac{d}{dx}\int_0^\delta u(U-u)\,dy = \frac{\tau_0}{\rho}$$

$$\frac{d}{dx}\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy = \frac{\tau_0}{\rho U^2}$$

$$\cancel{\frac{d}{dx}}\ \frac{\tau_0}{\rho U^2} = \frac{d}{dx}\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}\int_0^\delta \tanh\left(\frac{y}{\delta}\right)\left\{1-\tanh\left(\frac{y}{\delta}\right)\right\}dy$$

$$\frac{\tau_0}{\rho U^2} = \frac{d}{dx}\int_0^1 \tanh\eta\cdot(1-\tanh\eta)\,\delta\, d\eta \qquad \left(\text{let } \eta = \frac{y}{\delta}\right)$$

$$= \frac{d\delta}{dx}\cdot\int_0^1 \left[\tanh\eta - (1-\operatorname{sech}^2\eta)\right]d\eta$$

$$= \frac{d\delta}{dx}\cdot\left[\log\cosh\eta - \eta + \tanh\eta\right]_0^1$$

$$\frac{\tau_0}{\rho U^2} = \frac{d\delta}{dx} \cdot \left[\log\cosh 1 - \cancel{\log 1}^{\,0} - 1 + \tanh 1\right]$$

$$= \frac{d\delta}{dx} \cdot \left[0.433781 - 1 + 0.761594\right]$$

$$\frac{\tau_0}{\rho U^2} = \frac{d\delta}{dx} \cdot (0.1953752) \qquad \text{--- (1)}$$

now since

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$= \mu \left[\frac{\partial}{\partial y} U \tanh\left(\frac{y}{\delta}\right)\right]_{y=0}$$

$$= \mu U \left[\frac{1}{\delta} \operatorname{sech}^2\left(\frac{y}{\delta}\right)\right]_{y=0}$$

$$\tau_0 = \frac{\mu U}{\delta} \qquad \text{--- (2)}$$

By (1) & (2) we have

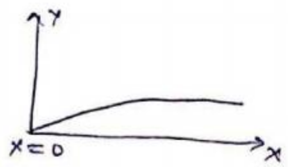
$$\frac{\left(\frac{\mu U}{\delta}\right)}{\rho U^2} = \frac{d\delta}{dx} \cdot (0.1953752)$$

$$\frac{\mu}{\rho \delta U} = \frac{d\delta}{dx} \, (0.1953752)$$

$$\frac{\nu}{U} = \delta \frac{d\delta}{dx} \cdot (0.1953752)$$

$$\delta \, d\delta = \frac{\nu}{U} \frac{dx}{(0.1953752)}$$

$$\frac{\delta^2}{2} = \frac{\nu}{U}(5.118358)\,x + C$$



When $x=0$, $\delta=0$ i.e $C=0$

So, $$\delta = \sqrt{2 \times 5.118358}\;\sqrt{\frac{\nu x}{U}}$$

$$\delta = 3.199487 \cdot \sqrt{\frac{\nu x}{U}}$$

$$\frac{d\delta}{dx} = \frac{3.199487}{2}\sqrt{\frac{\nu}{Ux}}$$

Substituting this value in Eqn(1)

$$\frac{\tau_0}{\rho U^2} = \frac{(0.1953752)(3.199487)}{2}\left(\frac{Ux}{\nu}\right)^{-\frac{1}{2}}$$

$$C_f = \frac{\tau_0}{\rho U^2} = 0.31255\left(\frac{Ux}{\nu}\right)^{-\frac{1}{2}}$$

skin friction coefficient

~~For velocity profiles are~~

For the following velocity profiles in the boundary layer

(i) $u = U\left[2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right]$

(ii) $u = U\sin\left\{\frac{\pi}{2}\left(\frac{y}{\delta}\right)\right\}$

Obtain the expression for δ_1, δ_2 and τ_0.